Simultaneous optimization

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A primer for discussion . . . Please send all comments/suggestions etc. so that I may include your contribution.

A. an equilibrium with nested flux surfaces

Let $\mathcal{S} \equiv \partial\mathcal{V}$ denote the toroidal boundary of volume, $\mathcal{V}$. Let $\mathbf{x}(\psi, \theta, \zeta)$ denote the "inverse" coordinate transformation, i.e. the coordinate transformation from straight fieldline flux coordinates to Cartesian, and assume that an arbitrary initial guess for the geometry of the internal flux surfaces is provided. The magnetic field is then defined by the coordinate transformation and the rotational-transform profile,

$$\mathbf{B} = \nabla\psi \times \nabla\theta + \iota\!\!\!-(\psi)\nabla\zeta \times \nabla\psi. \tag{1}$$

We may then consider the plasma energy functional to be function of both $\mathcal{S}$ and $\mathbf{x}$,

$$W[\mathbf{x}, \mathcal{S}] \equiv \int_{\mathcal{V}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv. \tag{2}$$

The typical equilibrium approach is to minimize W with respect to variations in the inverse coordinate mapping, $\mathbf{x}$, at fixed $\mathcal{S}$. We may use a descent algorithm.

Assuming ideal variations relate variation in the magnetic field and pressure to variations in the geometry of the flux surfaces, $\delta\mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$, $\delta p = (\gamma - 1)\, \boldsymbol{\xi} \cdot \nabla p - \gamma\nabla \cdot (p\,\boldsymbol{\xi})$, the first variation in W is

$$\delta W = \int_{\mathcal{V}} (\nabla p - \mathbf{j} \times \mathbf{B}) \cdot \boldsymbol{\xi}\, dv \;\; + \text{surface term}, \tag{3}$$

and so the descent algorithm is

$$\frac{\partial \mathbf{x}}{\partial \tau} = -(\nabla p - \mathbf{j} \times \mathbf{B}), \tag{4}$$

where τ is an integration parameter. The equilibrium must be considered as a function of $\mathcal{S}$.

As Rosenbluth explained, ideal variations are not analytic [1]. Ideal variations lead to overlapping flux surfaces at the rational rotational-transform surfaces.

B. a mixed ideal relaxed equilibrium

The first part of our task is to place the equilibrium calculation on a better mathmematical foundation. The plasma energy functional does not change, but the idea of multi-region relaxed MHD (MRxMHD) is to consider a different class of variations other than globally ideal. The suggestion of Hudson & Kraus [2] is to partition the plasma into finite volumes of ideal and relaxed plasmas. The plasma volume is partitioned into alternating ideal and relaxed regions, and mixed ideal-relaxed energy functional is

$$\mathcal{F} \equiv \sum_{i \in \mathcal{I}} \underbrace{\int_{\mathcal{V}_i} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{\delta\mathbf{B} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B})} + \sum_{j \in \mathcal{J}} \underbrace{\int_{\mathcal{V}_j} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{\int \mathbf{A}\cdot\mathbf{B}\, dv = H_j}. \tag{5}$$

In each ideal region, the topological constraints of ideal MHD are enforced; and to avoid the singularities in the parallel current the rotational-transform must be sufficiently irrational, e.g. $\mathbf{B} = \nabla\psi \times \nabla\theta - \iota\!\!\!-_i \nabla\psi \times \nabla\zeta$, where $\iota\!\!\!-_i$ is a noble irrational. Smooth pressure profiles may be supported. With the pressure and rotational-transform being constrained, the parallel current is only known aposteori.

In each relaxed region, arbitrary variations in the magnetic field are allowed subject to the constraint of conserved helicity in each volume (i.e., Taylor relaxation). The pressure is flat, the parallel current is constant, and the rotational-transform is apriori unknown (in fact, if the magnetic reconnection leads to irregular magnetic fieldlines, the rotational-transform may be undefined).

In the relaxed regions, the magnetic field is not frozen into the coordinates, and $\mathbf{B} = \nabla \times \mathbf{A}$ must be treated as separate independent quantities. The descent equations become

$$\frac{\partial \mathbf{x}}{\partial \tau} = \begin{cases} -(\nabla p - \mathbf{j} \times \mathbf{B}), & \text{in the ideal regions,} \\ -[[p + B^2/2]], & \text{at the interfaces,} \end{cases} \tag{6}$$

$$\frac{\partial \mathbf{A}}{\partial \tau} = -(\nabla \times \mathbf{B} - \mu \mathbf{B}), \quad \text{in the relaxed regions} \tag{7}$$

I believe the Robert MacKay wants to develop a more esoteric continuous model, where the pressure gradients are supported on a Cantor set; and this would be very interesting. The challenge is to not only develop a coherent mathematical formalism but to also develop a numerical discretization that can accommodate the structure of the solution.

C. optimization

The second part of our proposal is to address the optimization. A new approach is to consider the equilibrium and optimization problems simultaneously. Almost all previous fixed-boundary optimization strategies proceed by i) specify the boundary; ii) compute the equilibrium; iii) measure the relevant properties of the equilibrium; and then iv) change the shape of the boundary until the relevant properties improve. We might consider an approach that more intimately combines the optimization calculation with the equilibrium calculation.

Let $\mathcal{P}_i(\mathbf{B})$ be a heirarchy of "properties" of the magnetic field, such as flux-surface quality (size of islands), quasisymmetry, transport, complexity/cost of the external coil set for example; and that we wish to minimize each $\mathcal{P}_i$. The idea that there is a hierarchy reflects the fact that ensuring that there are no large overlapping islands and large volumes of irregular fieldlines must be considered a more important property of the magnetic field than neoclassical transport. We can augment the descent direction to include a equation that adjusts the geometry of the boundary to improve the plasma properties

$$\frac{\partial \mathbf{x}}{\partial \tau} = \begin{cases} -(\nabla p - \mathbf{j} \times \mathbf{B}), & \text{in the ideal regions,} \\ -[[p + B^2/2]], & \text{at the interfaces,} \end{cases} \tag{8}$$

$$\frac{\partial \mathbf{A}}{\partial \tau} = -(\nabla \times \mathbf{B} - \mu \mathbf{B}), \quad \text{in the relaxed regions} \tag{9}$$

$$\frac{\partial \mathbf{S}}{\partial \tau} = -\sum_i \omega_i \frac{\partial \mathcal{P}_i}{\partial \mathbf{B}} \cdot \frac{\partial \mathbf{B}}{\partial \mathcal{S}}, \tag{10}$$

During the equilibrium calculation itself, the *boundary* of the plasma is continuously being adjusted to improve the properties of interest, and ω_i are the user adjustable weights. This suggestion is reminiscent of the coil-healing algorithm used in the design of NCSX [3], in which the geometry of an external set of coils was adjusted calculation to cancel the formation of magnetic islands at every iteration of the equilibrium calculation. I believe that Harold Weitzner once made a suggestion along these lines; he said that adjusting the gometry of the boundary to eliminate islands/singularity should be considered as part of the equilibrium calculation, and he has some work along these lines [4, 5].

This requires that the plasma properties be a differentiable function of the magnetic field, and that the magnetic field is a differentiable function of the plasma boundary (which is why the above comments about non-analyticity of ideal-MHD is so important). In multi-region relaxed MHD, the magnetic field is a differentiable function of the boundary; in fact, the required derivatives have already been calculated.

We may also consider the following: if we are to modify the boundary so as to create a globally integrable magnetic field (as this is clearly preferable for magnetic confinement), then do we really need a code that can accommodate islands and chaos? Consider the NSTAB code (described in the review of 3D MHD [http://w3.pppl.gov/ shudson/Papers/Drafts/3DReview.pdf]. Garabedian explained that because of the conservation properties of the numerical discretization in NSTAB, that NSTAB could indicate the presence of islands by the formation of current sheets, even though the representation of the magnetic field assumed integrability. In this case, at every stage of the descent we would have an integrable field. The benefit of this is that most of the conventional measures of plasma properties, such as neoclassical transport, quasisymmetry even turbulent transport codes assume an integrable magnetic. If we could extract a measure of the islands in NSTAB, this would just be another property that we need to minimize, and we would give this a higher weight.

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